

# Geodesics in stationary spacetimes and classical Lagrangian systems<sup>☆</sup>

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Received 26 April 2006; revised 19 September 2006

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## Abstract

We state a fundamental correspondence between geodesics on stationary spacetimes and the equations of classical particles on Riemannian manifolds, accelerated by a potential and a magnetic field. By variational methods, we prove some existence and multiplicity theorems for fixed energy solutions (joining two points or periodic) of the above described Riemannian equation. As a consequence, we obtain existence and multiplicity results for geodesics with fixed energy, connecting a point to a line or periodic trajectories, in (standard) stationary spacetimes.

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MSC: 58E10; 53C50; 58E05; 70H40

Keywords: Lorentz metrics; Stationary spacetimes; Geodesics; Lagrangian systems; Lorentz force; Periodic trajectories

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## 1. Introduction

Aim of this article is to study the relation between geodesics on a certain class of Lorentzian manifolds and classical Lagrangian systems. More precisely:

- (a) we prove that spatial components of geodesics on (standard) stationary spacetimes solve the equations of classical particles moving on a Riemannian manifold, under the action of a potential and a magnetic field;

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<sup>☆</sup> Work supported by M.I.U.R. (research project “Metodi variazionali e topologici ed equazioni differenziali non lineari”).

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- (b) we prove existence and multiplicity of solutions with fixed total energy for the Riemannian equation in (a) (by means of a suitable variational principle extending the classical Maupertuis–Jacobi one);
- (c) we apply the results in (b) to prove existence and multiplicity of geodesics (joining a point to a line or periodic trajectories) with prescribed energy, on (standard) stationary spacetimes.

Before giving a detailed exposition of our results, we recall the basic notions of Lorentzian geometry which will be used throughout the paper (see e.g. [6,22]).

A Lorentzian manifold is called *stationary* if it admits a timelike Killing vector field. An important class of stationary Lorentzian manifolds is given by the (standard) stationary ones.

**Definition 1.** A Lorentzian manifold  $(L, \langle \cdot, \cdot \rangle_L)$  is a (standard) stationary spacetime if  $L = M \times \mathbb{R}$  is a product manifold,  $(M, \langle \cdot, \cdot \rangle)$  is any finite-dimensional, connected Riemannian manifold and the metric is given by

$$\langle \zeta, \zeta' \rangle_L = \langle \xi, \xi' \rangle + \langle \delta(x), \xi \rangle \tau' + \langle \delta(x), \xi' \rangle \tau - \beta(x) \tau \tau' \quad (1.1)$$

for any  $z = (x, t) \in L$ ,  $\zeta = (\xi, \tau)$ ,  $\zeta' = (\xi', \tau') \in T_z L = T_x M \times \mathbb{R}$ , where  $\delta$  and  $\beta$  are, respectively, a smooth vector field and a smooth strictly positive scalar field on  $M$ . If  $\delta \equiv 0$ , then  $L$  is called (standard) *static*.

Given a Lorentzian manifold  $(L, \langle \cdot, \cdot \rangle_L)$  and a geodesic  $z: I \rightarrow L$ ,  $I \subset \mathbb{R}$  interval,  $E_z \in \mathbb{R}$  exists such that

$$E_z = \frac{1}{2} \langle \dot{z}(s), \dot{z}(s) \rangle_L \quad \forall s \in I. \quad (1.2)$$

Throughout this paper  $E_z$  will be called “energy” (because of its relation to the energy of Lagrangian systems stated in Theorem 3).

As a vector  $\zeta \in TL$  is said *timelike* (respectively *lightlike*; *causal*; *spacelike*) if  $\langle \zeta, \zeta \rangle_L < 0$  (respectively  $\langle \zeta, \zeta \rangle_L = 0$ ,  $\zeta \neq 0$ ;  $\langle \zeta, \zeta \rangle_L \leq 0$ ;  $\langle \zeta, \zeta \rangle_L > 0$  or  $\zeta = 0$ ), by (1.2) a geodesic  $z$  is said to be timelike, lightlike, causal or spacelike according to the value of  $E_z$ .

When  $L$  is stationary a further conservation law holds. In the standard case,  $\partial t$  is a Killing vector field for  $\langle \cdot, \cdot \rangle_L$ , thus its product by a geodesic  $z = (x, t): I \rightarrow L$  is constant and  $K \in \mathbb{R}$  exists such that

$$\beta(x(s)) \dot{t}(s) - \langle \delta(x(s)), \dot{x}(s) \rangle = K \quad \forall s \in I. \quad (1.3)$$

We shall deal with geodesics having prescribed energy, both joining a fixed point to a line in  $L$  and  $t$ -periodic trajectories according to the following definition.

**Definition 2.** Given a (standard) stationary spacetime  $L$ , a  $t$ -periodic trajectory of universal period  $T$  and proper period  $a > 0$  is a geodesic  $z = (x, t): [0, a] \rightarrow L$  such that

$$\begin{cases} x(a) = x(0), \\ \dot{x}(a) = \dot{x}(0), \end{cases} \quad \begin{cases} t(a) = t(0) + T, \\ \dot{t}(a) = \dot{t}(0). \end{cases}$$

A  $t$ -periodic trajectory  $z = (x, t)$  is called *trivial* if  $x$  is a constant curve. On (standard) stationary spacetimes, a curve  $z(s) = (\bar{x}, t(s))$  is a (timelike) trivial periodic trajectory if and only if  $\bar{x}$  is a critical point of  $\beta$  and  $t(s) = Ts/a$  (see the geodesic equations (2.2)). If  $M$  is compact, trivial trajectories surely exist.

In the sequel we shall assume  $t(0) = 0$  and we call *geometrically distinct* two periodic trajectories having different ranges (which is equivalent to require that one of them cannot be obtained from the other by means of an affine parametrization). The choice  $t(0) = 0$  avoids obtaining trajectories having the same spatial components and with temporal components differing by a constant, that would be distinct according to the above definition but not interesting.

Geodesics on Lorentzian manifolds have been widely studied in last years, both by geometric and by variational methods, under assumptions about the growth of the metric coefficients and the topology of the underlying manifold. From a variational viewpoint, geodesics  $z: [a, b] \rightarrow L$  are critical points of the “energy” functional

$$f(z) = \int_a^b \langle \dot{z}, \dot{z} \rangle_L ds \quad (1.4)$$

with  $z$  varying in a suitable manifold of curves. Differently from the Riemannian case, functional  $f$  is not bounded from below so it is difficult to prove the existence of its critical points. When the metric coefficients do not depend explicitly on the time, as firstly done in [10] in the static case, it is possible to reduce the problem to the study of a purely Riemannian functional whose critical points are easier to find.

A different variational approach has been used in [25], where, in the particular case of a (standard) static spacetime  $L$ , the author proves that, if  $z = (x, t)$  is a geodesic on  $L$  such that (1.3) holds for some  $K \in \mathbb{R}$ , the component  $x$  is a solution of a Lagrangian system on  $M$  given by

$$D_s \dot{x} + \nabla V_K(x) = 0, \quad V_K = -\frac{K^2}{2\beta} \quad (1.5)$$

(where  $D_s$  denotes the covariant derivative along  $x$  with respect to the Levi-Civita connection and  $\nabla V$  is the gradient of  $V$  with respect to  $\langle \cdot, \cdot \rangle$ ). Vice versa, by each solution of (1.5) a static geodesic can be obtained. As Lagrangian systems like (1.5) are a classical topic in Riemannian geometry, the author obtains theorems for Lorentzian geodesics essentially as corollaries of Riemannian results. This method works for geodesics connecting a point to a line, periodic trajectories with fixed proper period or energy.

As far as we know, this approach has never been used in the more general (standard) stationary case with  $\delta \neq 0$  (see e.g. [13] for a study of geodesics on stationary spacetimes and applications to Kerr spacetime), because of the lack of a correspondence between geodesics and Lagrangian systems. The main result of this paper (Theorem 3) states that such a correspondence can be established again, endowing  $M$  by a perturbation of its natural metric  $\langle \cdot, \cdot \rangle$  and adding a term in Eq. (1.5), representing the action of a magnetic field.

More precisely, we shall denote by  $\langle \cdot, \cdot \rangle_1$  the Riemannian metric on  $M$  defined by

$$\langle \xi, \xi' \rangle_1 = \langle \xi, \xi' \rangle + \frac{1}{\beta(x)} \langle \delta(x), \xi \rangle \langle \delta(x), \xi' \rangle \quad (1.6)$$

for any  $x \in M$ ,  $\xi, \xi' \in T_x M$ . Note that this is a perturbation of the natural metric on  $M$  by means of the linear, self-adjoint, positive operator  $P(x)$ ,  $x \in M$ , on  $T_x M$  defined by

$$P(x)[\xi] = \frac{1}{\beta(x)} \langle \delta(x), \xi \rangle \delta(x) \quad \forall \xi \in T_x M$$

and it coincides with  $\langle \cdot, \cdot \rangle$  when  $M$  is static. From now on, we shall denote by  $\nabla^1$  the Levi-Civita connection of  $(M, \langle \cdot, \cdot \rangle_1)$  and by  $D_s^1$  the associated covariant derivative.

Let us consider a smooth potential and a smooth vector field on  $M$  given by

$$V(x) = -\frac{1}{\beta(x)}, \quad A(x) = \frac{\delta(x)}{\beta(x) + \langle \delta(x), \delta(x) \rangle}. \quad (1.7)$$

Let us denote by  $F^1$  the curl of  $A$ , that is the two-form defined by

$$F^1(X, Y) = \langle \nabla_X^1 A, Y \rangle_1 - \langle X, \nabla_Y^1 A \rangle_1 \quad (1.8)$$

for any smooth vector fields  $X, Y$  on  $M$ .

The suitable generalization of (1.5) is the class of differential equations (depending on a parameter  $K \in \mathbb{R}$ ) given by

$$D_s^1 \dot{x} + \frac{1}{2} K^2 \nabla^1 V(x) = K \hat{F}^1(x)[\dot{x}] \quad (1.9)$$

where  $\hat{F}^1 : TM \rightarrow TM$  is the linear map associated to  $F^1$  (that is  $F^1(x)[u, v] = \langle \hat{F}^1(x)[u], v \rangle_1$  for any  $x \in M$ ,  $u, v \in T_x M$ ). Equation (1.9) represents the motion of a classical particle on  $(M, \langle \cdot, \cdot \rangle_1)$  under the action of a conservative force having potential  $(K^2/2)V$  and a magnetic field described by  $KA$  (see (1.7)). Moreover, as  $\hat{F}^1$  is antisymmetric, each solution  $x : I \rightarrow M$ ,  $I \subset \mathbb{R}$  interval, of (1.9), has total energy given by

$$\frac{1}{2} \langle \dot{x}(s), \dot{x}(s) \rangle_1 + \frac{1}{2} K^2 V(x(s)) \quad \forall s \in I \quad (1.10)$$

(as in the case of (1.5) with null magnetic field).

A fundamental link between stationary geodesics and solutions of (1.9) is stated by the following theorem (for the proof see Section 2).

**Theorem 3.** *Let  $(L, \langle \cdot, \cdot \rangle_L)$  be a (standard) stationary spacetime (according to Definition 1) and  $I \subset \mathbb{R}$  an interval.*

*If  $z = (x, t) : I \rightarrow L$  is a geodesic verifying (1.3) for some  $K \in \mathbb{R}$ , then  $x : I \rightarrow \mathbb{R}$  solves (1.9).*

*Vice versa, a geodesic  $z = (x, t) : I \rightarrow L$  for  $\langle \cdot, \cdot \rangle_L$  can be obtained by a solution  $x : I \rightarrow M$  of (1.9) for some  $K \in \mathbb{R}$  and  $t$  verifying (1.3).*

*Moreover each particle  $x$  has total energy  $E$  (see (1.10)) equal to the energy  $\frac{1}{2} \langle \dot{z}, \dot{z} \rangle_L$  of the corresponding geodesic.*

Thus, also in the stationary case, solutions  $x : [0, a] \rightarrow M$  of (1.9) joining two fixed points  $x_0, x_1 \in M$  correspond to geodesics  $z : [0, a] \rightarrow L$  joining a point  $z_0 = (x_0, t_0)$  to a line  $(x_1, s) \subset L$ , while periodic solutions of (1.9) (i.e. smooth solutions  $x : [0, a] \rightarrow M$  such that

$x(0) = x(a)$ ,  $\dot{x}(0) = \dot{x}(a)$ ) give rise to periodic trajectories  $z: [0, a] \rightarrow L$  according to Definition 2.

For both boundary conditions, it is convenient to fix a value for the parameter  $K$  in (1.3). As in [25], we choose  $K = \sqrt{2}$  in order to normalize the coefficient of  $\nabla^1 V$  and obtain geometrically distinct trajectories (see comments after Theorem 5). This is not a restrictive choice because each geodesic  $z = (x, t)$  verifying (1.3) with a non-null value of  $K$  can be reparametrized in an unique way (using a homothety) to verify (1.3) with  $K = \sqrt{2}$ .

The simpler case, by a variational viewpoint, is that of geodesics connecting a point to a line, having a prescribed parametrization proportional to the arc length. They have a physical interpretation when  $E \leq 0$ . In the lightlike case, they represent the images of a source of light received by an observer, while in the timelike cases free falling massive particles, under the action of a gravitational field. In last years, they have been widely studied. Among the results on (standard) stationary manifolds, we recall [14] for the case  $E = 0$ , [5] where, in the static case, a wider range of energies ( $E \geq E_0$  where  $E_0$  is strictly negative, if  $\beta$  is bounded from above) is considered and [24] where the same problem is analyzed in the causal cases, by geometric methods. For further results in the causal cases, on more general classes of manifolds, we recall also [15–17].

Here we extend the results in [14], dealing with a larger interval of energies and the ones in [5], working on (standard) stationary spacetimes (as will be better clarified in Remark 9). More precisely, we shall prove in Section 3 the following theorem.

**Theorem 4.** *Let  $(L, \langle \cdot, \cdot \rangle_L)$  be a (standard) stationary spacetime (as in Definition 1) such that*

- (i)  $(M, \langle \cdot, \cdot \rangle)$  is a connected, complete, at least  $C^3$  Riemannian manifold;
- (ii)  $\bar{B} \in \mathbb{R}$  exists such that

$$\sup_{x \in M} \frac{|\delta(x)|}{\sqrt{\beta(x)(\beta(x) + \langle \delta(x), \delta(x) \rangle)}} = \bar{B}. \quad (1.11)$$

Then, for any  $E \in \mathbb{R}$  with

$$E > \bar{\beta} + \bar{B}^2 \quad \text{where } \bar{\beta} = \sup_{x \in M} \left( -\frac{1}{\beta(x)} \right) \quad (1.12)$$

and for any  $x_0, x_1 \in M$ ,  $x_0 \neq x_1$ ,  $t_0 \in \mathbb{R}$ , a geodesic  $z = (y, t): [0, a] \rightarrow L$  exists joining the point  $(x_0, t_0) \in L$  to the line  $(x_1, s) \subset L$ , such that

$$\frac{1}{2} \langle \dot{z}, \dot{z} \rangle_L = E \quad \text{and} \quad \beta(y) \dot{t} - \langle \delta(y), \dot{y} \rangle = \sqrt{2}. \quad (1.13)$$

If  $M$  is not contractible in itself, a sequence  $(z_m)$ ,  $z_m = (y_m, t_m): [0, a_m] \rightarrow M$  of geodesics as above exists. The arrival times  $t_m(a_m)$  verify

$$\lim_{m \rightarrow +\infty} t_m(a_m) = +\infty \quad (1.14)$$

when  $\beta$  is bounded from above and, denoted by

$$\bar{B}_1 = \sup_{x \in M} \frac{|\delta(x)|}{\sqrt{\beta(x) + \langle \delta(x), \delta(x) \rangle}}, \quad N = \sup_{x \in M} \beta(x),$$

when

- $\bar{B}_1 < \sqrt{EN + 1}$  for any possible  $E \leq 0$ ;
- $\bar{B}_1 < 1/\sqrt{EN + 1}$  for any  $E > 0$ .

Condition (ii) is equivalent to require the boundedness of the vector field  $A$  with respect to the norm induced by  $\langle \cdot, \cdot \rangle_1$ , as we shall discuss in Section 2, together with simpler conditions implying (ii) ( $\beta$  bounded from below or  $\delta/\beta$  bounded).

The sign of the energy  $E$  in Theorem 4 depends on the sign of the constant in the right-hand side of (1.12). It may occur that it is strictly negative (in such a case  $\beta$  is bounded from above) so that an interval of strictly negative and all positive energies are allowed. Otherwise, if it is positive, an unbounded interval of strictly positive energies can be considered.

Now, we can deal with applications to  $t$ -periodic trajectories. We remark that this class of geodesics is physically relevant, because causal ones are the relativistic version of periodic motions (under a gravitational force) in classical Lagrangian mechanics. They were firstly introduced in [8], for static spacetimes with  $M = \mathbb{R}^3$ . In this reference and in [10], existence of timelike trajectories is proved for static spacetimes with  $M = \mathbb{R}^n$ . On (standard) stationary spacetimes with compact  $M$ , we recall papers [19] (for timelike trajectories with fixed universal period, in the static case) and [26] (for timelike trajectories obtained by geometric methods). In both papers the fundamental group of  $M$  is assumed non-trivial. In the more difficult case of non-compact  $M$ , different techniques have been used. We recall the already mentioned [25] and [2–4] where assumptions about the sectional curvature of  $M$  at infinity are imposed. In [9], in the static case, the existence of a function on  $M$  convex at infinity is postulated and in [11,21] this approach has been extended to the (standard) stationary case, respectively for null energy and fixing the universal period.

Our next theorem (proved in Section 2) extends the results of these last papers, as will be discussed in details in Remark 10. Before stating it, we introduce a new metric  $\langle \cdot, \cdot \rangle_{1,E}$  associated to  $\langle \cdot, \cdot \rangle_1$  (see (1.6))

$$\langle \xi, \xi \rangle_{1,E} = \left( E + \frac{1}{\beta(x)} \right) \langle \xi, \xi \rangle_1 \quad \forall x \in M, \xi \in T_x M, \quad (1.15)$$

well defined on  $M$  if  $E > \bar{\beta}$ .

**Theorem 5.** *Let  $(L, \langle \cdot, \cdot \rangle_L)$  be a (standard) stationary spacetime (as in Definition 1) such that  $M$  verifies (i) of Theorem 4 and*

- (iii)  *$M$  is not contractible in itself and its fundamental group  $\pi_1(M)$  is finite or it has infinitely many conjugacy classes.*

*Assume also that, for some  $x_0 \in M$*

$$(iv) \quad \lim_{d(x,x_0) \rightarrow +\infty} \frac{|\delta(x)|}{\sqrt{\beta(x)(\beta(x) + \langle \delta(x), \delta(x) \rangle)}} = 0. \quad (1.16)$$

*Consider  $E \in \mathbb{R}$  such that (1.12) holds and*

(v)  $U \in C^2(M, \mathbb{R})$  and  $R, \mu > 0$  exist such that, for any  $x \in M$  with  $d(x, x_0) \geq R$

$$H_{1,E}^U(x)[\xi, \xi] \geq \mu \langle \xi, \xi \rangle_{1,E} \quad \forall \xi \in T_x M$$

where  $H_{1,E}^U(x)[\xi, \xi]$  is the Hessian of  $U$  with respect to  $\langle \cdot, \cdot \rangle_{1,E}$  (see (1.15)) at  $x$  in the direction of  $\xi$ ;

(vi) denoted by  $|\cdot|_*$  the norm of endomorphisms on  $T_x M$  induced by  $\langle \cdot, \cdot \rangle$

$$\lim_{d(x, x_0) \rightarrow +\infty} \left| \nabla U(x) \right| \left| \nabla \left( \frac{\delta}{\beta} \right)(x) \right|_* = 0.$$

Then, one non-trivial  $t$ -periodic trajectory  $z = (y, t) : [0, a] \rightarrow L$  exists such that (1.13) holds.

Due to the periodicity, assumptions of Theorem 5 are stronger than the ones in Theorem 4. Precisely, the topological assumption (ii) has been introduced (as in the quoted papers on this topic) and (iv) which reinforces (ii). Condition (v) concerns the existence of a function  $U$  on  $M$  convex with respect to (1.15). In Section 2 we shall suggest sufficient conditions for (v) and classes of functions verifying it. Finally, (vi) controls the behavior of the gradient of  $U$  at infinity with respect to the differential of  $\delta/\beta$ .

We remark that Theorem 5 works for the same range of energies of Theorem 4, thus the same considerations about the causal character of geodesics hold in this case. Moreover, fixing the energy  $E$  and the value  $K = \sqrt{2}$  for the constant in (1.3) (see (1.13)) is the right choice in order to get a multiplicity result: periodic trajectories obtained by Theorem 5 for different values of the energy  $E$  are geometrically distinct.

Now, it remains to deal with the Riemannian results. Periodic orbits under the action of a magnetic flow as in Eq. (1.9) have been studied e.g. in [1,7,18] on compact manifolds, when  $V = 0$ . Here, using variational methods, we find conditions under which a differential equation like (1.9) on a non-compact manifold admits solutions (joining two points or periodic) with fixed energy. Due to the physical meaning of (1.9), the obtained results are interesting in themselves (besides their application to Lorentzian geodesics), so we present them below.

Given a Riemannian manifold  $(M, \langle \cdot, \cdot \rangle)$ , a smooth function  $V : M \rightarrow \mathbb{R}$  and an exact two-form  $F$  on  $M$ , a general equation like (1.9) takes the form

$$D_s \dot{x} + \nabla V(x) = \hat{F}(x)[\dot{x}] \quad (1.17)$$

where  $D_s$  denotes the covariant derivative induced by the Levi-Civita connection,  $\nabla V$  is the gradient of  $V$  with respect to  $\langle \cdot, \cdot \rangle$  and  $\hat{F} : TM \rightarrow TM$  is the linear map associated to  $F$ . Let  $\omega(x) = \langle A(x), \cdot \rangle$  be the one-form such that  $d\omega = F$ ,  $A$  a smooth vector field on  $M$ .

As already observed, the energy of a solution of (1.17)  $x : I \rightarrow M$ ,  $I \subset \mathbb{R}$  interval, is a constant  $E_x \in \mathbb{R}$  such that

$$E_x = \frac{1}{2} \langle \dot{x}(s), \dot{x}(s) \rangle + V(x(s)) \quad \forall s \in I. \quad (1.18)$$

The Maupertuis–Jacobi principle states that, when  $F$  is null, solutions of (1.17) with fixed energy  $E$ , are, up to reparametrizations, geodesics with respect to a Jacobi metric

$$\langle \xi, \xi \rangle_E = (E - V(x)) \langle \xi, \xi \rangle \quad (1.19)$$

for any  $x \in M$ ,  $\xi \in T_x M$ , which is well defined in a neighborhood of  $x \in M$  if  $E > V(x)$ . We refer to [20] for an historical review of this classical principle and its precise and simpler formulation. In Section 3 we shall show how to generalize the geodesic equation with respect to  $\langle \cdot, \cdot \rangle_E$ , when  $F \neq 0$ , in order to obtain curves which, suitably reparametrized, solve (1.17) and have total energy equal to  $E$  (see Eq. (3.1) and Proposition 11). The generalized equation has a variational structure. In order to get critical points of its associated functional, we assume that

- (M<sub>1</sub>)  $(M, \langle \cdot, \cdot \rangle)$  is a connected, complete, at least  $C^3$  Riemannian manifold;
- (H<sub>1</sub>)  $V \in C^1(M, \mathbb{R})$  is bounded from above;
- (H<sub>2</sub>)  $F$  is an exact two-form on  $M$ ,  $F = d\omega$  for a  $C^1$  one-form  $\omega$ , whose  $\langle \cdot, \cdot \rangle$ -associated vector field  $A$  is bounded.

By (H<sub>1</sub>)–(H<sub>2</sub>) the following constants

$$\bar{V} = \sup_{x \in M} V(x), \quad \bar{A} = \sup_{x \in M} |A(x)| \quad (1.20)$$

are well defined ( $|\cdot|$  denotes the norm on  $T_x M$  induced by  $\langle \cdot, \cdot \rangle$ ). We shall deal with energies  $E$  such that

$$E > \bar{V} + \frac{\bar{A}^2}{2}. \quad (1.21)$$

The previous assumptions are sufficient to obtain our first result (see Section 4 for the proof), concerning solutions joining two fixed points.

**Theorem 6.** *Let  $(M, \langle \cdot, \cdot \rangle)$  be a Riemannian manifold such that (M<sub>1</sub>) holds,  $V : M \rightarrow \mathbb{R}$  a potential on  $M$  satisfying (H<sub>1</sub>) and  $F$  a two-form on  $M$  satisfying (H<sub>2</sub>). Let  $E \in \mathbb{R}$  verify (1.21). Then, for any  $x_0, x_1 \in M$ ,  $x_0 \neq x_1$ , the following statements hold:*

- (a) *a solution  $y : [0, a] \rightarrow M$  of (1.17) exists such that  $y(0) = x_0$  and  $y(a) = x_1$  having energy  $E$ ;*
- (b) *if  $M$  is not contractible in itself, a sequence  $(y_m)$ ,  $y_m : [0, a_m] \rightarrow M$  of solutions of (1.17) as in (a) exists, such that, denoted by  $l(y_m)$  their lengths, it is*

$$\lim_{m \rightarrow +\infty} l(y_m) = +\infty. \quad (1.22)$$

In our setting, it is  $E > \bar{V}$ , thus the Jacobi metric (1.19) is well defined on  $M$  and is complete, if (M<sub>1</sub>) holds. The stronger inequality (1.21) is necessary to ensure that the functional involved in the proof of Theorem 6 is coercive.

The study of periodic trajectories of (1.17) is a more difficult problem which requires (when  $M$  is not compact) stronger assumptions to control the behavior at infinity of our problem's data.

Besides (M<sub>1</sub>),  $M$  has to verify the following topological assumption:

- (M<sub>2</sub>)  $M$  is not contractible in itself and its fundamental group  $\pi_1(M)$  is finite or it has infinitely many conjugacy classes.



Instead of (H<sub>2</sub>), we assume that for some  $x_0 \in M$

(H<sub>3</sub>)  $F$  is an exact two-form on  $M$ ,  $F = d\omega$  for a  $C^1$  one-form  $\omega$  whose  $\langle \cdot, \cdot \rangle$ -associated vector field  $A$  verifies

$$\lim_{d(x, x_0) \rightarrow +\infty} |A(x)| = 0$$

(where  $d$  is the distance induced by  $\langle \cdot, \cdot \rangle$ ).

When  $M$  is not bounded, we follow the technique introduced in [9], based on the existence of a function  $U$  on  $M$  convex at infinity. Since we are dealing with a fixed energy problem, we assume that  $U$  is convex at infinity with respect to the Jacobi metric (while in [9] this convexity holds with respect to  $\langle \cdot, \cdot \rangle$ ). Thus, if  $E > \bar{V}$ , we assume that

(H<sub>4</sub>)  $U \in C^2(M, \mathbb{R})$  and  $R, \mu > 0$  exist such that, for any  $x \in M$  with  $d_E(x, x_0) \geq R$ ,

$$H_E^U(x)[\xi, \xi] \geq \mu \langle \xi, \xi \rangle_E \quad \forall \xi \in T_x M$$

where  $d_E$  is the distance induced by  $\langle \cdot, \cdot \rangle_E$  and  $H_E^U(x)[\xi, \xi]$  the Hessian of  $U$  with respect to  $\langle \cdot, \cdot \rangle_E$  at  $x$  in the direction of  $\xi$ .

Finally, next assumption links the behavior of function  $U$  and vector field  $A$  at infinity. Precisely,

(H<sub>5</sub>) denoted by  $|\cdot|_*$  the norm of endomorphisms on  $T_x M$  induced by  $\langle \cdot, \cdot \rangle$ ,

$$\lim_{d(x, x_0) \rightarrow +\infty} |\nabla U(x)| |\nabla A(x)|_* = 0.$$

Our main result is the following theorem (proved in Section 4).

**Theorem 7.** *Let  $(M, \langle \cdot, \cdot \rangle)$  be a Riemannian manifold such that (M<sub>1</sub>)–(M<sub>2</sub>) hold,  $V : M \rightarrow \mathbb{R}$  a potential on  $M$  satisfying (H<sub>1</sub>) and  $F$  a two-form on  $M$  satisfying (H<sub>3</sub>). If (H<sub>4</sub>)–(H<sub>5</sub>) hold for  $x_0 \in M$  and  $E$  verifying (1.21), then at least one non-constant periodic solution  $y : [0, a] \rightarrow M$  of (1.17) exists having energy  $E$ .*

We shall discuss in details assumption (H<sub>4</sub>) in Remark 21 comparing it with the one in [9].

Using standard arguments it is possible to prove the existence of infinitely many solutions of (1.17) with the same energy  $E$ . The form of (1.17) and the presence of the field  $F$  allow us to claim that they are generically geometrically distinct.

Finally, the proof of Theorem 7 will clarify that, if  $M$  is compact, assumptions (H<sub>3</sub>)–(H<sub>5</sub>) are unnecessary and the thesis of Theorem 7 hold under (H<sub>1</sub>)–(H<sub>2</sub>) and (M<sub>1</sub>)–(M<sub>2</sub>). As a consequence, when  $M$  is compact Theorem 5 holds under (i)–(iii).

## 2. Geodesics on stationary Lorentzian manifolds

In this section we prove Theorem 3 and then the existence of geodesics on (standard) stationary spacetimes (Theorems 4, 5). Moreover we compare these results with previous ones on this topic, discussing the role of the hypothesis.

It is useful to write the equation of a geodesic  $z = (x, t): [a, b] \rightarrow L$  using metric  $\langle \cdot, \cdot \rangle_1$  (see (1.6)) instead of the natural metric on  $M$ . It is a critical point of the functional  $f$  defined by (1.4) on the manifold of the  $H^1$ -curves joining  $z(a)$  to  $z(b)$ . Tangent vectors at  $z$  are the  $H^1$ -vector fields  $\zeta = (\xi, \tau)$  along  $z$  such that  $\zeta(a) = 0 = \zeta(b)$  (see also Section 3 for a more detailed description of these manifolds of curves). By (1.6), it is easy to obtain the following equality:

$$\langle \delta(x), \xi \rangle = \beta(x) \langle A(x), \xi \rangle_1 \quad \forall x \in M, \xi \in T_x M \quad (2.1)$$

(where  $A(x)$  is the vector field defined in (1.7)). Thus (1.1), (2.1) and straightforward calculations allow one to write

$$f(z) = f(x, t) = \int_a^b (\langle \dot{x}, \dot{x} \rangle_1 - \beta(x) (i - \langle A(x), \dot{x} \rangle_1)^2) ds.$$

For any  $C_0^\infty$ -vector field  $\zeta = (\xi, \tau)$  along  $z$ , if  $z$  is a geodesic, we obtain, integrating by parts,

$$\begin{aligned} 0 &= df(z)[\zeta] \\ &= \int_a^b (2\langle \dot{x}, \dot{\xi} \rangle_1 - \langle \nabla^1 \beta(x), \xi \rangle_1 (i - \langle A(x), \dot{x} \rangle_1)^2 \\ &\quad - 2\beta(x) (i - \langle A(x), \dot{x} \rangle_1) (\dot{i} - \langle \nabla_\xi^1 A(x), \dot{x} \rangle_1 - \langle A(x), \dot{\xi} \rangle_1)) ds \\ &= \int_a^b \left( -2\langle D_s^1 \dot{x}, \xi \rangle_1 - \langle \nabla^1 \beta(x), \xi \rangle_1 (i - \langle A(x), \dot{x} \rangle_1)^2 \right. \\ &\quad \left. - 2\beta(x) (i - \langle A(x), \dot{x} \rangle_1) \dot{i} + 2\beta(x) (i - \langle A(x), \dot{x} \rangle_1) \langle \hat{F}^1(x)[\dot{x}], \xi \rangle_1 \right. \\ &\quad \left. - 2 \frac{d}{ds} (\beta(x) (i - \langle A(x), \dot{x} \rangle_1)) \langle A(x), \xi \rangle_1 \right) ds. \end{aligned}$$

Taking  $\xi = 0$  and  $\tau = 0$  in the previous equation gives the differential equations verified by  $z$ , i.e.

$$\begin{cases} \frac{d}{ds} (\beta(x) (i - \langle A(x), \dot{x} \rangle_1)) = 0, \\ D_s^1 \dot{x} + \frac{1}{2} (i - \langle A(x), \dot{x} \rangle_1)^2 \nabla^1 \beta(x) = \beta(x) (i - \langle A(x), \dot{x} \rangle_1) \hat{F}^1(x)[\dot{x}]. \end{cases} \quad (2.2)$$

Vice versa, each smooth curve  $z: (x, t): [a, b] \rightarrow L$  verifying (2.2) is a geodesic.

**Remark 8.** By Eq. (2.2), a curve  $z = (x, t): [a, b] \rightarrow L$  is a geodesic such that

$$i = \frac{\langle \delta(x), \dot{x} \rangle}{\beta(x)} = \langle A(x), \dot{x} \rangle_1$$

(i.e. (1.3) holds for  $K = 0$ ) if and only if  $x$  is a geodesic for  $\langle \cdot, \cdot \rangle_1$ . In analogy to the static case (see [25]), these ones can be considered as basic geodesics. Our choice for the constant  $K$  ensures that in Theorems 4 and 5 we deal with non-basic geodesics.

Now it is easy to prove Theorem 3.

**Proof of Theorem 3.** A smooth curve  $z = (x, t) : I \rightarrow L$ ,  $I \subset \mathbb{R}$  interval, is a geodesic, if and only if it solves (2.2). The first equation in (2.2) gives (1.3) for some  $K \in \mathbb{R}$  (see (2.1) to get the equality) and the second one gives (1.9). Moreover, by (2.1) and (2.2), it is

$$\frac{1}{2} \langle \dot{z}(s), \dot{z}(s) \rangle_L = \frac{1}{2} \langle \dot{x}(s), \dot{x}(s) \rangle_1 - \frac{1}{2} \frac{K^2}{\beta(x(s))}$$

for any  $s \in I$  and the proof is complete.  $\square$

Theorems 4 and 5 are a direct application of Theorems 6 and 7, respectively, once we take  $K = \sqrt{2}$  in order to normalize the coefficient of  $\nabla^1 V$  in (1.9), that is we consider equation

$$D_s^1 \dot{x} + \nabla^1 V(x) = \sqrt{2} \hat{F}^1(x)[\dot{x}] \quad (2.3)$$

where  $V$  and  $A$  have been defined in (1.7),  $F^1$  in (1.8) and  $\hat{F}^1$  is the linear map associated to  $F^1$ . By a solution  $x : [0, a] \rightarrow M$  of (2.3) having fixed energy  $E$ , defining

$$t(s) = t_0 + \int_0^s \frac{\sqrt{2} + \langle \delta(x), \dot{x} \rangle}{\beta(x)} d\tau \quad (2.4)$$

we obtain

- (a) a geodesic  $z = (x, t) : [0, a] \rightarrow L$  connecting a point  $(x_0, t_0)$  to the line  $(x_1, s)$ , if  $x$  joins  $x_0$  to  $x_1$  in  $M$ ;
- (b) a periodic trajectory  $z = (x, t) : [0, a] \rightarrow L$  according to Definition 2, if  $x$  is a periodic solution of (2.3),

in both cases with energy  $E$ . In (b) we choose  $t_0 = 0$  in (2.4), in order to avoid trivial trajectories. Thus the universal period is given by

$$T = \int_0^a \frac{\sqrt{2} + \langle \delta(x), \dot{x} \rangle}{\beta(x)} ds.$$

**Proof of Theorem 4.** From previous remarks, we only need to verify that all the assumptions of Theorem 6 hold for Eq. (2.3).

We re-formulated  $(M_1)$  in (i) of Theorem 4 for metric  $\langle \cdot, \cdot \rangle$ , which is complete if and only if  $\langle \cdot, \cdot \rangle_1$  is complete. Indeed, by the Cauchy–Schwartz inequality, we get

$$\langle \xi, \xi \rangle \leq \langle \xi, \xi \rangle_1 \leq \left( 1 + \frac{\langle \delta(x), \delta(x) \rangle}{\beta(x)} \right) \langle \xi, \xi \rangle \quad \forall x \in M, \quad \xi \in T_x M,$$

so  $\langle \cdot, \cdot \rangle$  and  $\langle \cdot, \cdot \rangle_1$  are locally equivalent.

Assumption  $(H_1)$  is verified because  $\beta$  is strictly positive ( $\bar{\beta}$  in (1.12) is negative).

For  $(H_2)$ , we have to ensure that the vector field  $\sqrt{2}A$  is bounded with respect to  $|\cdot|_1$ . By (1.6), it is not difficult to obtain

$$|A(x)|_1 = \frac{|\delta(x)|}{\sqrt{\beta(x)(\beta(x) + \langle \delta(x), \delta(x) \rangle)}} \quad (2.5)$$

which is imposed to be bounded in (ii). As in this case  $\bar{A} = \sqrt{2}\bar{B}$ , condition (1.21) becomes (1.12).

To complete the proof, it remains to justify (1.14). By Proposition 11, the sequence  $(z_m(s) = (y_m(s), t_m(s)))_m$ ,  $s \in [0, a_m]$  of geodesics is chosen in way that each  $y_m$  is a reparametrization of a curve  $x_m : [0, 1] \rightarrow M$ . Precisely,  $y_m(s) = x_m(r_m(s))$  where

$$\begin{aligned} \dot{r}_m(s) &= \sqrt{\frac{2}{c_m}} \left( E + \frac{1}{\beta(x_m(s))} \right), \\ c_m &= \left( E + \frac{1}{\beta(x_m(s))} \right) \langle \dot{x}_m(s), \dot{x}_m(s) \rangle_1 \end{aligned} \quad (2.6)$$

and

$$\lim_{m \rightarrow +\infty} c_m = +\infty. \quad (2.7)$$

The arrival times  $t_m(a_m)$ , by (2.1), (2.4) and (2.6), are equal to

$$\begin{aligned} t_m(a_m) &= t_0 + \int_0^{a_m} \left( \frac{\sqrt{2}}{\beta(y_m)} + \langle A(y_m), \dot{y}_m \rangle_1 \right) ds \\ &= t_0 + \int_0^1 \left( \frac{\sqrt{c_m}}{\beta(x_m)E + 1} + \langle A(x_m), \dot{x}_m \rangle_1 \right) ds. \end{aligned}$$

Thus, the Cauchy–Schwartz inequality and the definition of  $A$  give

$$t_m(a_m) \geq t_0 + \sqrt{c_m} \int_0^1 \left( \frac{1}{\beta(x_m)E + 1} - \frac{\bar{B}_1}{\sqrt{\beta(x_m)E + 1}} \right) ds \quad (2.8)$$

so that, by (2.7), the sequence  $(t_m(a_m))_m$  diverges in the cases listed in Theorem 4.  $\square$

We observe that sufficient conditions for (ii) are, for example, one among the following:

$$\text{a positive number } \nu \text{ exists such that } \beta(x) \geq \nu \text{ for any } x \in M; \quad (2.9)$$

$$\text{a positive number } C \text{ exists such that } \frac{|\delta(x)|}{\beta(x)} \leq C \text{ for any } x \in M. \quad (2.10)$$

Moreover from (2.8), it is easy to check that, if  $\beta$  verifies (2.9), the sequence  $(t_m(a_m))$  diverges for any possible  $E < 0$  without further conditions.

**Remark 9.** In [14], on a spacetime  $L = M \times \mathbb{R}$  endowed with a metric  $\langle \cdot, \cdot \rangle_L$  conformal to a (standard) stationary one, i.e.

$$\langle \zeta, \zeta \rangle_L = \alpha(z) (\langle \xi, \xi \rangle + 2\langle \delta(x), \xi \rangle \tau - \beta(x) \tau^2)$$

for any  $z = (x, t) \in L$ ,  $\zeta = (\xi, \tau) \in T_z L = T_x M \times \mathbb{R}$  (where  $\delta$  and  $\beta$  are as in Definition 1 and  $\alpha$  is a smooth, strictly positive scalar field on  $M$ ), the authors prove existence (and also multiplicity, when  $M$  is not contractible in itself) of lightlike geodesics joining a point to a line. They assume that

- (a)  $(M, \frac{1}{\beta(x)} \langle \cdot, \cdot \rangle)$  is complete;
- (b)  $\sup_{x \in M} \frac{\langle \delta(x), \delta(x) \rangle}{\beta(x)}$  is finite.

This result can be re-obtained by Theorem 4. Indeed, as lightlike geodesics, up to reparametrizations, are invariant by conformal changes of metric, it is possible to apply Theorem 4 to  $L$  with the (standard) stationary metric

$$\langle \zeta, \zeta \rangle_{L,1} = \langle \xi, \xi \rangle_2 + 2\langle \delta(x), \xi \rangle_2 \tau - \tau^2, \quad \langle \xi, \xi \rangle_2 = \frac{\langle \xi, \xi \rangle}{\beta(x)}$$

for any  $z = (x, t) \in L$ ,  $\zeta = (\xi, \tau) \in T_z L$ . So, we need to assume that  $(M, \langle \cdot, \cdot \rangle_2)$  is complete (i.e. (a)) while condition (ii) is certainly verified as

$$\bar{B} = \sup_{x \in M} \frac{|\delta(x)|_2}{\sqrt{1 + \langle \delta(x), \delta(x) \rangle_2}} = \sup_{x \in M} \frac{|\delta(x)|}{\sqrt{\beta(x) + \langle \delta(x), \delta(x) \rangle}} \leq 1.$$

Our theorem works for energies  $E > -1 + \bar{B}$  including  $E = 0$  if we impose  $\bar{B} < 1$ , which is equivalent to (b).

Moreover, we point out that applying Theorem 4 to (standard) static spacetimes, one obtains Proposition 5.1 of [5]. In particular, the estimates about the cases in which the arrival times diverge extend the ones in that paper.

**Proof of Theorem 5.** We begin by observing that, as  $\langle \cdot, \cdot \rangle$  and  $\langle \cdot, \cdot \rangle_1$  are locally equivalent, in (iv)–(vi) we have used  $d$ , instead of the distance associated to  $\langle \cdot, \cdot \rangle_1$ , obtaining conditions equivalent to (H<sub>3</sub>)–(H<sub>5</sub>) of Theorem 7.

Thus, (iii) and (iv) are an application of (M<sub>2</sub>) and (H<sub>3</sub>), respectively. Condition (v) is (H<sub>4</sub>).

It remains to discuss (vi). It is a reformulation of (H<sub>5</sub>) that should be written

$$\lim_{d(x, x_0) \rightarrow +\infty} |\nabla^1 U(x)|_1 |\nabla^1 A(x)|_{*,1} = 0$$

where  $|\cdot|_1$  is the norm on  $T_x M$  and  $|\cdot|_{*,1}$  is the norm of endomorphisms, induced by  $\langle \cdot, \cdot \rangle_1$ . By (2.1), it is easy to obtain

$$(I + P(x))[\nabla^1 A(x)] = \nabla \left( \frac{\delta}{\beta} \right)(x) \quad \forall x \in M.$$

Thus, (vi) immediately follows by this equality after observing that, as  $P(x)$  is a positive operator, it is

$$|\nabla^1 U(x)|_1 \leq |\nabla U(x)| \quad (2.11)$$

for any  $x \in M$ .  $\square$

We end this section with some remarks about the assumptions of Theorem 5. Instead of (v), one could impose conditions depending only on the original metric of  $M$ , obtained by writing the relations linking the Hessians of  $U$  with respect to the different metrics involved in this problem. Firstly, the Hessians with respect to  $\langle \cdot, \cdot \rangle_{1,E}$  and  $\langle \cdot, \cdot \rangle_1$  verify

$$\begin{aligned} H_{1,E}^U(x)[\xi, \xi] &= H_1^U(x)[\xi, \xi] \\ &+ \frac{2\langle \nabla^1 U(x), \xi \rangle_1 \langle \nabla^1 \beta(x), \xi \rangle_1 - \langle \nabla^1 U(x), \nabla^1 \beta(x) \rangle_1 \langle \xi, \xi \rangle_1}{2\beta^2(x)(E + \frac{1}{\beta(x)})} \end{aligned}$$

for any  $x \in M$ ,  $\xi \in T_x M$  (see (4.19) for  $V = -1/\beta$ ). Thus, a first, immediate sufficient condition for (v) is

$$\begin{aligned} U \text{ is } \langle \cdot, \cdot \rangle_1\text{-convex at infinity, } \beta \text{ verifies (2.9),} \\ \lim_{d(x, x_0) \rightarrow +\infty} |\nabla^1 U(x)|_1 |\nabla^1 \beta(x)|_1 = 0. \end{aligned} \quad (2.12)$$

Moreover, one can evaluate the  $\langle \cdot, \cdot \rangle_1$ -Hessian of  $U$  in terms of the  $\langle \cdot, \cdot \rangle$ -Hessian, giving, for any  $x \in M$ ,  $\xi \in T_x M$ ,

$$\begin{aligned} H_1^U(x)[\xi, \xi] &= H^U(x)[\xi, \xi] + \frac{\langle \delta(x), \xi \rangle}{\beta(x)} \langle \nabla U(x), B(x)[\hat{\delta}(x)[\xi]] \rangle \\ &- \frac{\langle \delta(x), \xi \rangle^2}{2\beta^2(x)} \langle \nabla U(x), B(x)[\nabla \beta(x)] \rangle \\ &+ \frac{\langle \delta(x), \xi \rangle \langle \nabla \beta(x), \xi \rangle}{\beta^2(x)} \langle \nabla U(x), B(x)[\delta(x)] \rangle \\ &- \frac{\langle \nabla_\xi \delta(x), \xi \rangle}{\beta(x)} \langle \nabla U(x), B(x)[\delta(x)] \rangle, \end{aligned} \quad (2.13)$$

where  $B(x)$  is the inverse operator of  $P(x)$  and  $\hat{\delta}(x)$  is the linear map on  $T_x M$  associated to the curl of  $\delta$ . Thus, as  $|B(x)|_* \leq 1$ , by (2.13) and (2.11), a sufficient condition for (2.12) (and so for (v)) is

$$\begin{aligned} U \text{ is } \langle \cdot, \cdot \rangle\text{-convex at infinity, (2.9) and (2.10) hold,} \\ \lim_{d(x, x_0) \rightarrow +\infty} |\nabla U(x)| |\nabla \beta(x)| = 0, \quad \lim_{d(x, x_0) \rightarrow +\infty} |\nabla U(x)| |\nabla \delta(x)|_* = 0. \end{aligned} \quad (2.14)$$

We also point out that, as

$$\nabla\left(\frac{\delta}{\beta}\right)(x) = \frac{\nabla\delta(x)}{\beta(x)} - \langle \nabla\beta(x), \cdot \rangle \frac{\delta(x)}{\beta^2(x)} \quad \forall x \in M,$$

sufficient conditions for (vi) are (2.9), (2.10) and the two limits in (2.14).

Thus, (2.14) allows one to obtain classes of functions satisfying (v) and (vi). For example, taking  $L = \mathbb{R}^N \times \mathbb{R}$  (endowing  $\mathbb{R}^N$  with the usual Euclidean metric) and a stationary metric on  $L$  with constant  $\delta$  and  $\beta$ , each function  $U : \mathbb{R}^N \rightarrow \mathbb{R}$  strictly convex at infinity verifies (v) and (vi).

**Remark 10.** In [11], the existence of at least one lightlike periodic trajectory is proved on (standard) stationary spacetimes verifying (a) of Remark 9, the topological assumptions (i) and (iii) of Theorem 5 and (using the same notations of Remark 9)

$$\lim_{d(x,x_0) \rightarrow +\infty} |\delta(x)|_2 = 0, \quad \lim_{d(x,x_0) \rightarrow +\infty} |\nabla U(x)|_2 |\nabla\delta(x)|_{*,2} = 0, \quad (2.15)$$

where  $|\cdot|_{*,2}$  is the norm of endomorphisms induced by  $\langle \cdot, \cdot \rangle_2$  and  $U$  is a function  $\langle \cdot, \cdot \rangle_2$ -convex at infinity. We observe that the first limit in (2.15) is equivalent to (iv) (written for  $\langle \cdot, \cdot \rangle_2$ ) while the second one, together with the convexity of  $U$ , imply (v) and (vi). Thus, also this result can be re-obtained by Theorem 5.

Finally, we observe that our assumptions are weaker than the ones in [21], where the existence of timelike periodic trajectories, with fixed universal period, on (standard) stationary spacetimes has been proved, if (iii), the existence of a function  $U$   $\langle \cdot, \cdot \rangle$ -convex at infinity, (2.9), (2.10), the two limits in (2.14) and further hypotheses hold.

### 3. The variational framework for the Riemannian results

When in Eq. (1.17)  $F = 0$ , the classical Maupertuis–Jacobi principle ensures that solutions with fixed energy  $E$  correspond (up to reparametrizations) to geodesics with respect to the Jacobi metric  $\langle \cdot, \cdot \rangle_E$  defined at (1.19) (see also [20] for a detailed description of this principle). Here we shall deal with its extension to the case when a magnetic field acts (Proposition 11).

Assuming that  $(M_1)$ ,  $(H_1)$ – $(H_2)$  hold and taking  $E \in \mathbb{R}$ ,  $E > \bar{V}$  (see (1.20)), let us consider the following differential equation

$$(E - V(x))D_s^E \dot{x} = \sqrt{\frac{1}{2} \langle \dot{x}, \dot{x} \rangle_E} \hat{F}(x)[\dot{x}] \quad (3.1)$$

where  $D_s^E$  denotes the covariant derivative with respect to  $\langle \cdot, \cdot \rangle_E$ .

We observe that, differently from (1.17), (3.1) is invariant by affine reparametrizations  $as + b$  if  $a \geq 0$ , hence it is not restrictive to take into account solutions defined on the interval  $[0, 1]$ .

**Proposition 11.** *If  $x \in C^2([0, 1], M)$  is a non-constant solution of (3.1), then  $a > 0$  and a reparametrization  $y \in C^2([0, a], M)$  of  $x$  exist, solving*

$$\begin{cases} D_s \dot{y} + \nabla V(y) = \hat{F}(y)[\dot{y}], \\ \frac{1}{2} \langle \dot{y}, \dot{y} \rangle + V(y) = E \end{cases} \quad (3.2)$$

and vice versa.

**Proof.** Let  $x : [0, 1] \rightarrow M$  be a non-constant solution of (3.1). As  $\hat{F}$  is antisymmetric, contracting both sides of (3.1) by  $\dot{x}$  gives the existence of  $c_x > 0$  such that

$$\langle \dot{x}(s), \dot{x}(s) \rangle_E = c_x \quad \forall s \in [0, 1]. \quad (3.3)$$

Then, by (3.3) and the relation between the covariant derivatives with respect to two conformal metrics, (3.1) can be written as

$$D_s((E - V(x))\dot{x}) + \frac{1}{2}\langle \dot{x}, \dot{x} \rangle \nabla V(x) = \sqrt{\frac{c_x}{2}} \hat{F}(x)[\dot{x}]. \quad (3.4)$$

Let us consider the diffeomorphism  $\alpha : [0, 1] \rightarrow [0, a]$  defined by

$$\alpha(s) = \sqrt{\frac{c_x}{2}} \int_0^s \frac{d\tau}{E - V(x(\tau))} \quad \forall s \in [0, 1]$$

where

$$a = \sqrt{\frac{c_x}{2}} \int_0^1 \frac{d\tau}{E - V(x(\tau))}.$$

Let  $y : [0, a] \rightarrow M$  be defined by  $y(s) = x(\alpha^{-1}(s))$ ,  $s \in [0, a]$ . As

$$\dot{y} = \sqrt{\frac{2}{c_x}} (E - V(x(\alpha^{-1}))) \dot{x}(\alpha^{-1}),$$

(3.4) and easy calculations allow one to prove that  $y$  verifies (3.2).

In a similar way, each solution of (3.2)  $y(s)$ ,  $s \in [0, a]$ , is non-constant and can be reparametrized to a solution  $x$  of (3.1) such that  $\langle \dot{x}(s), \dot{x}(s) \rangle_E = 1$ , by means of the diffeomorphism

$$\beta(s) = \int_0^s (E - V(y(\tau))) d\tau$$

for any  $s \in [0, a]$ .  $\square$

In order to describe the variational structure of (3.1) we need to define some manifolds of curves. As  $(M_1)$  holds, by the Nash embedding theorem, we can assume that  $M$  is a submanifold of an Euclidean space  $\mathbb{R}^N$  and  $\langle \cdot, \cdot \rangle$  is restriction to  $M$  of the usual Euclidean metric. Thus, we can identify the infinite-dimensional Sobolev manifold  $H^1([0, 1], M)$  with the set of absolutely continuous curves  $x : [0, 1] \rightarrow \mathbb{R}^N$  with square summable derivative, such that  $x([0, 1]) \subset M$ . As  $(M, \langle \cdot, \cdot \rangle)$  is complete, also  $H^1([0, 1], M)$  is complete with respect to the Riemannian structure

$$\langle \xi, \xi \rangle = \int_0^1 \langle \xi, \xi \rangle ds + \int_0^1 \langle D_s \xi, D_s \xi \rangle ds$$



for any  $x \in H^1([0, 1], M)$  and  $\xi \in T_x H^1([0, 1], M)$ . In order to deal with the two different boundary conditions of our problem, we consider two smooth submanifolds of  $H^1([0, 1], M)$  given by

$$\Omega^1(x_0, x_1, M) = \{x \in H^1([0, 1], M) \mid x(0) = x_0, x(1) = x_1\}$$

for some fixed  $x_0, x_1$  in  $M$ ,  $x_0 \neq x_1$ , and

$$\Lambda^1(M) = \{x \in H^1([0, 1], M) \mid x(0) = x(1)\}.$$

They are complete and for any  $x \in \Omega^1(x_0, x_1, M)$  (or respectively  $x \in \Lambda^1(M)$ ) the tangent spaces at  $x$  are given by

$$T_x \Omega^1(x_0, x_1, M) = \{\xi \in T_x H^1([0, 1], M) \mid \xi(0) = 0 = \xi(1)\},$$

$$T_x \Lambda^1(M) = \{\xi \in T_x H^1([0, 1], M) \mid \xi(0) = \xi(1)\}.$$

At first, let us consider the fixed extreme points case, introducing the functional  $G_1 : \Omega^1(x_0, x_1, M) \rightarrow \mathbb{R}$  defined by

$$G_1(x) = \sqrt{2 \int_0^1 \langle \dot{x}, \dot{x} \rangle_E ds} + \int_0^1 \langle A(x), \dot{x} \rangle ds \quad (3.5)$$

for any  $x \in \Omega^1(x_0, x_1, M)$ . Note that, as  $V$  is smooth and the two metrics on  $M$  are locally equivalent, the first integral in (3.5) is well defined when  $x \in \Omega^1(x_0, x_1, M)$ . Standard calculations show that  $G_1$  is smooth and the following proposition holds.

**Proposition 12.** *Let  $x \in \Omega^1(x_0, x_1, M)$  be a critical point of  $G_1$ . Then  $x$  is a  $C^2$  curve solving (3.1).*

In next section we shall prove that, if (1.21) holds, critical points of  $G_1$  really exist.

When we deal with periodic solutions, we have to consider the same functional on  $\Lambda^1(M)$ , i.e.  $G_2 : \Lambda^1(M) \rightarrow \mathbb{R}$  defined as in (3.5) for any  $x \in \Lambda^1(M)$ . Note that  $G_2$  is not differentiable on any constant curve in  $\Lambda^1(M)$ . Nevertheless, it is easy to prove the following proposition.

**Proposition 13.** *Let  $x \in \Lambda^1(M)$  be a non-constant curve, critical point of  $G_2$ . Then  $x$  is of class  $C^2$ , solves (3.1) and  $\dot{x}(0) = \dot{x}(1)$ .*

We shall prove in Section 4 that critical points of  $G_2$  exist, under assumptions  $(M_2)$ ,  $(H_3)$ – $(H_5)$ .

#### 4. Proof of Theorems 6 and 7

Here, applying the results in Section 3, we study critical points of functionals  $G_1$  and  $G_2$ . In both cases, abstract tools based on the notion of Ljusternik–Schnirelman category will be used (for more details see e.g. [23]).

**Definition 14.** Let  $X$  be a topological space. The *Ljusternik–Schnirelman category* of a subset  $A$  of  $X$ , briefly  $\text{cat}_X(A)$ , is the least number of closed and contractible subsets of  $X$  covering  $A$ . If  $A$  cannot be covered by a finite number of such sets, it is  $\text{cat}_X(A) = +\infty$ .

In the sequel we shall use this notation:

$$\text{cat}(X) = \text{cat}_X(X).$$

Existence of critical points of  $G_1$  relies on the following theorem.

**Theorem 15.** Let  $\Omega$  be a complete Riemannian manifold and  $G$  a  $C^1$  functional on  $\Omega$ , bounded from below and satisfying the Palais–Smale condition at level  $a$  for any  $a \geq \inf_{x \in \Omega} G(x)$  (i.e. any  $(x_m)_m \subset \Omega$  such that

$$\lim_{m \rightarrow +\infty} G(x_m) = a, \quad \lim_{m \rightarrow +\infty} G'(x_m) = 0 \quad (4.1)$$

converges in  $\Omega$  up to subsequences). Then  $G$  has a minimum point. Moreover if  $\text{cat}(\Omega) = +\infty$ , a sequence  $(x_m)_m$  of critical points of  $J$  exists such that

$$\lim_{m \rightarrow +\infty} G(x_m) = +\infty.$$

The following result (proved in [12]) is a sufficient condition in order to ensure that the manifold  $\Omega^1(x_0, x_1, M)$  verifies the last assumption of Theorem 15.

**Proposition 16.** If  $M$  is a non-contractible in itself Riemannian manifold, for any  $x_0, x_1 \in M$

$$\text{cat}(\Omega^1(x_0, x_1, M)) = +\infty$$

and  $\Omega^1(x_0, x_1, M)$  contains compact subsets of arbitrary large category.

**Proof of Theorem 6.** Assumptions  $(H_1)$ – $(H_2)$  and inequality (1.21) guarantee that  $G_1$  is bounded from below. Indeed, by the Cauchy–Schwartz inequality, it is

$$\left| \int_0^1 \langle A(x), \dot{x} \rangle ds \right| \leq \frac{\bar{A}}{\sqrt{E - \bar{V}}} \left( \int_0^1 \langle \dot{x}, \dot{x} \rangle_E ds \right)^{1/2} \quad \forall x \in \Omega^1(x_0, x_1, M) \quad (4.2)$$

where  $\bar{V}$ ,  $\bar{A}$  are as in (1.20). Thus for any  $x \in \Omega^1(x_0, x_1, M)$  it is

$$G_1(x) \geq \left( \sqrt{2} - \frac{\bar{A}}{\sqrt{E - \bar{V}}} \right) \left( \int_0^1 \langle \dot{x}, \dot{x} \rangle_E ds \right)^{1/2} \quad (4.3)$$

and the coefficient of the integral in (4.3) is positive, when (1.21) holds. Standard arguments and (4.3) allow one to prove that  $G_1$  verifies the Palais–Smale condition at any level greater than its infimum (see e.g. [14, Proposition 3.4]).

Applying Theorem 15 to  $G_1$ , let  $\bar{x} \in \Omega^1(x_0, x_1, M)$  be a minimum point of  $G_1$  on  $\Omega^1(x_0, x_1, M)$ . By Propositions 11 and 12,  $\bar{x}$  can be reparametrized to a solution of (3.2) so part (a) of Theorem 6 is proved.

Part (b) is a consequence of Proposition 16, the second part of Theorem 15 and the variational principles (Propositions 11 and 12). To complete the proof, let us consider a sequence  $(x_m)_m$  of critical points of  $G_1$  such that

$$\lim_{m \rightarrow +\infty} G_1(x_m) = +\infty.$$

By Proposition 11, the corresponding solutions  $y_m$  of (1.17) are defined in  $[0, a_m]$  where

$$a_m = \sqrt{\frac{c_{x_m}}{2}} \int_0^1 \frac{ds}{E - V(x_m)}, \quad c_{x_m} = \langle \dot{x}_m(s), \dot{x}_m(s) \rangle_E$$

and, denoted by  $l(y_m)$  the length of  $y_m$ , it is

$$l(y_m) = \sqrt{c_{x_m}} \int_0^1 \frac{ds}{\sqrt{E - V(x_m)}}.$$

As  $E - V$  is bounded from above on compact subsets of  $M$  and, by (4.2),

$$G_1(x_m) \leq \left( \sqrt{2} + \frac{\bar{A}}{\sqrt{E - \bar{V}}} \right) \sqrt{c_{x_m}},$$

(1.22) follows.  $\square$

Theorem 7 cannot be proved by searching directly critical points of functional  $G_2$ . Indeed, when  $M$  is not compact, the periodicity of the problem prevents  $G_2$  from satisfying the Palais–Smale condition. In order to overcome this problem we use a penalization technique (introduced in [9]) based on the existence of a function  $U$  convex at infinity (see (H<sub>4</sub>)).

From now on, we assume that (M<sub>1</sub>)–(M<sub>2</sub>), (H<sub>1</sub>), (H<sub>3</sub>)–(H<sub>5</sub>) hold. For any  $\varepsilon > 0$  we consider smooth functions  $\psi_\varepsilon : [0, +\infty[ \rightarrow \mathbb{R}$  defined by

$$\psi_\varepsilon(s) = \begin{cases} 0, & 0 \leq s \leq 1/\varepsilon, \\ \sum_{n=3}^{+\infty} \frac{1}{n!} (s - \frac{1}{\varepsilon})^n, & s > 1/\varepsilon. \end{cases}$$

Notice that two positive constants  $a_\varepsilon, b_\varepsilon$  exist such that

$$\psi'_\varepsilon(s) \geq \psi_\varepsilon(s) \geq a_\varepsilon s - b_\varepsilon \quad \forall s \geq 0. \quad (4.4)$$

We consider a family of penalized functionals for  $G_2$ , that is we define, for any  $\varepsilon > 0$ ,  $G_{2,\varepsilon} : A^1(M) \rightarrow \mathbb{R}$  by

$$G_{2,\varepsilon}(x) = \sqrt{2} \int_0^1 \langle \dot{x}, \dot{x} \rangle_E ds + 2 \int_0^1 \psi_\varepsilon(U(x)) ds + \int_0^1 \langle A(x), \dot{x} \rangle ds$$

for any  $x \in \Lambda^1(M)$ , where  $U$  has been introduced in (H<sub>4</sub>).

Our aim is to prove that  $G_{2,\varepsilon}$  satisfies the Palais–Smale condition and its critical points are critical points of  $G_2$ , when  $\varepsilon$  is sufficiently small. It will be useful the following lemma proved in [9, Lemma 2.2].

**Lemma 17.** *Let  $U, x_0, \mu$  be as in assumption (H<sub>4</sub>). Then  $C_1, C_2, C_3 > 0$  exist such that, denoted by  $\nabla^E$  the gradient with respect to  $\langle \cdot, \cdot \rangle_E$ , for any  $x \in M$  it is*

$$\langle \nabla^E U(x), \nabla^E U(x) \rangle_E^{1/2} \geq \mu d_E(x, x_0) - C_1, \quad (4.5)$$

$$U(x) \geq \frac{\mu}{2} d_E^2(x, x_0) - C_2 d_E(x, x_0) - C_3. \quad (4.6)$$

In order to get critical points of  $G_{2,\varepsilon}$  we shall apply the following abstract theorem.

**Theorem 18.** *Let  $\Lambda$  be a complete Riemannian manifold. Let  $G$  be a  $C^1$  functional, bounded from below and satisfying the Palais–Smale condition at level  $a$  for any  $a \geq \inf_{x \in \Lambda} G(x)$ . For any  $m \in \mathbb{N} \setminus \{0\}$ , let us define*

$$c_m = \inf_{A \in \Gamma_m} \sup_{x \in A} G(x), \quad \Gamma_m = \{A \subset \Lambda \mid \text{cat}_\Lambda(A) \geq m\}. \quad (4.7)$$

Then, if  $\Gamma_m$  is not empty and  $c_m$  is finite,  $c_m$  is a critical value of  $G$ .

We observe that, reasoning as in the proof of (4.3),  $D_0 > 0$  exists such that

$$G_{2,\varepsilon}(x) \geq G_2(x) \geq D_0 \left( \int_0^1 \langle \dot{x}, \dot{x} \rangle_E ds \right)^{1/2} \quad \forall x \in \Lambda^1(M) \quad (4.8)$$

so that  $G_{2,\varepsilon}$  is bounded from below and its minimum is 0 (attained on each constant curve of  $\Lambda^1(M)$ ). Even if  $G_{2,\varepsilon}$  is not differentiable on constant curves, Theorem 18 can still be applied to strictly positive values of  $c_m$ .

The crucial point of the proof is to obtain a priori estimates for critical points of  $G_{2,\varepsilon}$ . Each non-constant critical point  $x \in \Lambda^1(M)$  of  $G_{2,\varepsilon}$  is a  $C^2$  curve such that  $\dot{x}(0) = \dot{x}(1)$ , solving the differential equation

$$(E - V(x)) D_s^E \dot{x} = \psi'_\varepsilon(U(x)) \nabla U(x) + f_\varepsilon(x) \hat{F}(x) [\dot{x}] \quad (4.9)$$

where

$$f_\varepsilon(x) = \sqrt{\frac{1}{2} \int_0^1 \langle \dot{x}, \dot{x} \rangle_E ds + \int_0^1 \psi_\varepsilon(U(x)) ds}. \quad (4.10)$$

Hence, we can prove the following proposition.

**Proposition 19.** Let  $M, K \in \mathbb{R}$  be two positive numbers. Then,  $\varepsilon_1 > 0$  and  $D > 0$  exist such that, for any  $\varepsilon \in ]0, \varepsilon_1[$  and for any critical point  $x$  of  $G_{2,\varepsilon}$  verifying

$$K \leq G_{2,\varepsilon}(x) \leq M \quad (4.11)$$

it is

$$\sup_{s \in [0,1]} d_E(x(s), x_0) \leq D. \quad (4.12)$$

**Proof.** If, by contradiction, (4.12) does not hold, we could find an infinitesimal sequence  $(\varepsilon_m)_m$  and a sequence  $(x_m)_m$  of critical points of  $G_{2,m} \equiv G_{2,\varepsilon_m}$  verifying (4.11) and

$$\lim_{m \rightarrow +\infty} \sup_{s \in [0,1]} d_E(x_m(s), x_0) = +\infty.$$

By (4.8) and (4.11), as  $d$  and  $d_E$  are locally equivalent, also

$$\lim_{m \rightarrow +\infty} \inf_{s \in [0,1]} d(x_m(s), x_0) = +\infty. \quad (4.13)$$

Setting  $u_m(s) = U(x_m(s))$ ,  $s \in [0, 1]$  and  $\psi_m = \psi_{\varepsilon_m}$ , by (4.9) and the relation

$$(E - V(x))\nabla^E U(x) = \nabla U(x)$$

we obtain

$$\begin{aligned} \ddot{u}_m(s) &= H_E^U(x_m(s))[\dot{x}_m(s), \dot{x}_m(s)] + \langle D_s^E \dot{x}_m(s), \nabla^E U(x_m(s)) \rangle_E \\ &= H_E^U(x_m(s))[\dot{x}_m(s), \dot{x}_m(s)] \\ &\quad + \psi'_m(U(x_m(s))) \langle \nabla^E U(x_m(s)), \nabla^E U(x_m(s)) \rangle_E \\ &\quad + f_m(x_m) \langle \nabla^E U(x_m(s)), \hat{F}(x_m(s))[\dot{x}_m(s)] \rangle \end{aligned} \quad (4.14)$$

where  $f_m \equiv f_{\varepsilon_m}$  (see (4.10)). Hence, as  $\dot{u}_m(0) = \dot{u}_m(1)$ , by (H<sub>4</sub>) and (4.14) it is

$$\begin{aligned} 0 &= \int_0^1 \ddot{u}_m(s) \\ &\geq \mu \int_0^1 \langle \dot{x}_m, \dot{x}_m \rangle_E ds + \int_0^1 \psi'_m(U(x_m)) \langle \nabla^E U(x_m), \nabla^E U(x_m) \rangle_E ds \\ &\quad + \int_0^1 \frac{f_m(x_m)}{E - V(x_m)} \langle \nabla U(x_m), \hat{F}(x_m)[\dot{x}_m] \rangle ds. \end{aligned} \quad (4.15)$$

By (4.11) and (4.2), as

$$G_{2,m}(x_m) = 2f_m(x_m) + \int_0^1 \langle A(x_m), \dot{x}_m \rangle ds$$

it is not difficult to prove that  $(f_m(x_m))_m$  is a bounded sequence. Hence, by (4.13) and  $(H_5)$ , the last integral in (4.15) is an infinitesimal sequence. Moreover, (4.11), (4.13) and  $(H_3)$  imply that

$$\int_0^1 \langle \dot{x}_m, \dot{x}_m \rangle_E ds \geq D_1 + d_m - 2 \int_0^1 \psi_m(U(x_m)) ds \quad (4.16)$$

where  $D_1 > 0$  and  $(d_m)_m$  is an infinitesimal sequence. Then, (4.15) and (4.16) give

$$0 \geq \mu D_1 + g_m + \int_0^1 (\psi'_m(U(x_m)) \langle \nabla^E U(x_m), \nabla^E U(x_m) \rangle_E - 2\mu \psi_m(U(x_m))) ds$$

where  $(g_m)_m$  is an infinitesimal sequence. The integral in the last formula is positive, because the first inequality in (4.4) holds and, by (4.5) and (4.13), for  $m$  sufficiently large

$$\langle \nabla^E U(x_m(s)), \nabla^E U(x_m(s)) \rangle_E \geq 2\mu \quad \forall s \in [0, 1].$$

Thus, we obtain

$$0 \geq \mu D_1 + g_m$$

which is a contradiction.  $\square$

To complete our proof, we recall the following result contained in [12].

**Proposition 20.** *If  $(M_2)$  holds, then*

$$\text{cat}(\Lambda^1(M)) = +\infty$$

*and  $\Lambda^1(M)$  contains compact sets of arbitrary large category.*

**Proof of Theorem 7.** The choice of the penalization term and (4.6), by slight modifications of [11, Lemma 4.1], allow one to state that, for any  $\varepsilon > 0$ ,  $G_{2,\varepsilon}$  satisfies the Palais–Smale condition at each  $a > 0$ . Moreover, following [9,11], it is easy to prove that, for any  $a > 0$ , the sublevels

$$G_2^a = \{x \in \Lambda^1(M) \mid G_2(x) \leq a\}$$

have finite category. Hence we can deduce that, for any  $a > 0$ ,  $k_0 \in \mathbb{N}$  exists such that

$$B \cap (G_2)_a \neq \emptyset \quad \forall B \in \Gamma_{k_0}, \quad (G_2)_a = \{x \in \Lambda^1(M) \mid G_2(x) > a\}$$

and  $\Gamma_{k_0}$  defined in (4.7). For any  $\varepsilon \in ]0, 1]$  and  $B \in \Gamma_{k_0}$  it is

$$0 < a \leq \sup_{x \in B} G_2(x) \leq \sup_{x \in B} G_{2,\varepsilon}(x) \leq \sup_{x \in B} G_{2,1}(x)$$

which implies

$$a \leq c_{k_0,\varepsilon} \leq c_{k_0,1} \leq M_1, \quad M_1 = \max_{x \in K} G_{2,1}(x)$$

and  $K$  compact subset of  $\Lambda^1(M)$  of category larger than  $k_0$  (whose existence is given by Proposition 20). Then  $c_{k_0,\varepsilon}$  is a critical value of  $G_{2,\varepsilon}$  (see Theorem 18) satisfying an inequality like (4.11). By Proposition 19, if  $\varepsilon$  is sufficiently small,  $c_{k_0,\varepsilon}$  is also a critical point of  $G_2$  that, after a reparametrization (see Propositions 11 and 13), gives a periodic solution of (1.17) with energy  $E$ .  $\square$

**Remark 21.** It is easy to verify that Proposition 19 holds also when, instead of  $(H_4)$ , we assume that

$$U \in C^2(M, \mathbb{R}), \quad R, \mu > 0 \text{ exist such that, for any } x \in M \text{ with } d(x, x_0) \geq R$$

$$H^U(x)[\xi, \xi] \geq \mu \langle \xi, \xi \rangle \quad \forall \xi \in T_x M, \quad (4.17)$$

$$V \text{ is bounded from below and } \lim_{d(x, x_0) \rightarrow +\infty} |\nabla U(x)| |\nabla V(x)| = 0. \quad (4.18)$$

On the other hand, as the relation between the Hessians of the two conformal metrics  $\langle \cdot, \cdot \rangle$  and  $\langle \cdot, \cdot \rangle_E$  is given by

$$\begin{aligned} H_E^U(x)[\xi, \xi] &= H^U(x)[\xi, \xi] \\ &+ \frac{2\langle \nabla U(x), \xi \rangle \langle \nabla V(x), \xi \rangle - \langle \nabla U(x), \nabla V(x) \rangle \langle \xi, \xi \rangle}{2(E - V(x))} \end{aligned} \quad (4.19)$$

for any  $x \in M$ ,  $\xi \in T_x M$ , (4.17) and (4.18) imply our assumption  $(H_4)$ . Vice versa,  $(H_4)$  and the limit in (4.18) imply the  $\langle \cdot, \cdot \rangle$ -convexity of  $U$  at infinity. By simple examples (using one variable functions), one can easily verify that, if the limit in (4.18) is not 0,  $U$  can be  $\langle \cdot, \cdot \rangle$ -convex but not  $\langle \cdot, \cdot \rangle_E$ -convex at infinity, even if  $V$  is bounded from below (take e.g.  $U$  such that  $U' = x$  and  $V(x) = -\sin x$  for  $E > 1$ ).

Thus, we have chosen to impose  $(H_4)$ , avoiding a further condition at infinity (the limit in (4.18)).

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